Solution Sheet 7

Exercise 7.1

Let $F : \mathcal{X} \times \mathcal{Y} \to \mathcal{X}$ be continuous and bounded, (ξ_n) a collection of i.i.d $\mathcal{Y}$-valued random variables independent of the $\mathcal{X}$-valued random variable X_0. Define

$$X_{n+1} = F(X_n, \xi_n).$$

Prove that the Markov Process X induces a Feller semigroup.

Proof. Set $f \in C_b(\mathcal{X})$, then

$$\begin{aligned} Tf(x) &= \mathbb{E}(f(X_{n+1})|X_n = x) \\ &= \mathbb{E}(f(F(x, \xi_n)) \\ &= \int_\Omega f(F(x, \xi_n) d\mathbb{P} \\ &= \int_\mathcal{X} f(F(x, y))(\xi_n^* \mathbb{P})(dy). \end{aligned}$$

Set μ as the law of ξ_n, which is independent of n as the variables are i.i.d. Take now an arbitrary sequence $x_k \to x$ to test continuity. We see that

$$\begin{aligned} \lim_{k\to\infty} Tf(x_k) &= \lim_{k\to\infty} \int_\mathcal{X} f(F(x_k, y))\mu(dy) \\ &= \int_\mathcal{X} \lim_{k\to\infty} f(F(x_k, y))\mu(dy) \\ &= \int_\mathcal{X} f(F(x, y))\mu(dy) \\ &= Tf(x) \end{aligned}$$

having applied the dominated convergence theorem, noting that f is bounded and this supremum is integrable with respect to the probability measure μ. We of course also used the continuity of both f and F in its first variable, hence of the composition.

□

Exercise 7.2

Define the transition probability $P(x, A) = \int_A \phi(y)dy$ if $x \geq 0$ and $P(x, A) = \int_A \psi(y)dy$ if $x < 0$. Give conditions such that the associated semigroup is Feller. Provide an example where this semigroup is not Feller.

Proof. A simple condition is that for every $f \in C_b(\mathcal{X})$,

$$\int_\mathcal{X} f(y)\phi(y)dy = \int_\mathcal{X} f(y)\psi(y)dy.$$

This is due to the fact that

$$Tf(x) = \int_\mathcal{X} f(y)P(x, dy)$$

which is constant except for the singular jump at $x = 0$, hence this is the only possible point of discontinuity. The condition ensures that Tf remains constant. As an example where the Feller property is not satisfied, take $\mathcal{X} = [-1, 1]$ with ϕ and ψ supported on $[0, 1]$ defined by $\phi(y) = 2y$, $\psi(y) = 3y^2$. Then for $f(x) = x$, $Tf(x) = \frac{2}{3}$ if $x \geq 0$ and $\frac{3}{4}$ if $x < 0$, hence Tf is discontinuous at zero.

□

Exercise 7.3

Define the translation semigroup

$$(T_t f)(x) = f(x + t).$$

1. Verify the semigroup property and that T_t is a linear isometry on $C_0(\mathbb{R})$ i.e. $\|T_t\| = 1$.
2. Show that (T_t) is strongly continuous on $C_0(\mathbb{R})$.
3. Show that (T_t) is strongly continuous and $\|T_t\| = 1$ on $L^p(\mathbb{R})$ with $1 \leq p < \infty$.
4. Identify the generator of (T_t).
5. Show that (T_t)i s not be strongly continuous on $L^\infty(\mathbb{R})$.

Proof.

1. The semigroup property is clear. For the isometry,

$$sup_{f:\|f\|=1} \|T_t f\| = sup_{f:\|f\|=1} \|f\| = 1.$$

2. For a given $f \in C_0(\mathbb{R})$ and $\varepsilon > 0$, we can take a compact set K such that $|f(x)| \leq \varepsilon$ outside of K, noting also that f is uniformly continuous on K, from which strong continuity is deduced.

3. $\|T_t\| = 1$ is again a direct consequence of the transformation formula for integrals and the shift-invariance of the Lebesgue measure

$$sup_{f:\|f\|=1} \left(\int_{\mathbb{R}} |T_t f|^p dx \right)^{\frac{1}{p}} = sup_{f:\|f\|=1} \left(\int_{\mathbb{R}} |f|^p dx \right)^{\frac{1}{p}} = 1.$$

The strong continuity is a consequence of the dominated convergence theorem and the density of C_0 in L^p. We note that

$$\left(\int_{\mathbb{R}} |(T_t - 1)f|^p dx \right)^{\frac{1}{p}} \leq \left(\int_{\mathbb{R}} 2|f|^p dx \right)^{\frac{1}{p}},$$

approximate f by continuous functions g_n and use dominated convergence,

$$\left(\int_{\mathbb{R}} |(T_t - 1)f|^p dx \right)^{\frac{1}{p}} \leq \left(\int_{\mathbb{R}} |(T_t)(f - g_n)|^p dx \right)^{\frac{1}{p}} + \left(\int_{\mathbb{R}} |(T_t - 1)g_n|^p dx \right)^{\frac{1}{p}} + \left(\int_{\mathbb{R}} |(f - g_n)|^p dx \right)^{\frac{1}{p}}.$$

The only problematic term is the middle one, for which convergence holds when restricted to compact sets as in the previous part and the tails must be uniformly small for large n.

4. We simply identify the generator through a pointwise limit, for any given $x \in \mathbb{R}$,

$$\lim_{t \to 0} \frac{T_t f(x) - f(x)}{t} = \lim_{t \to 0} \frac{f(x+t) - f(x)}{t} = \frac{d}{dx} f(x)$$

hence the generator, at least formally, is given by $\frac{d}{dx}$. We leave the norm convergence of this quantity, dependent on the chosen norm, up to the interested student.

5. $$\left\| T(t) 1_{[0,1]} - 1_{[0,1]} \right\|_{L^\infty(\mathbb{R})} = 1$$

for all $t \neq 0$.

□

Exercise 7.4

Let T_t be a strongly continuous semigroup on a Banach Space E. Prove that there exists constants $M \geq 1$, $k \geq 0$ such that $\|T_t\| \leq M e^{kt}$. *Hint: Recall the Uniform Boundedness Principle, that for a collection of continuous linear operators $S_i : \mathcal{X} \to \mathcal{Y}$ where $\mathcal{X}$ is a Banach Space, if for every $x \in \mathcal{X}$ we have that* $\sup_i \|S_i(x)\|_{\mathcal{Y}} < \infty$ *then* $\sup_i \|S_i\| < \infty$ *for the operator norm.*

Proof. We first show that there exists an $a > 0, M \geq 1$ such that $\sup_{t \in [0,a]} \|T_t\| \leq M$. Suppose not for a contradiction, then there exists a sequence $t_n \to 0$ such that $\|T_{t_n}\| \to \infty$. Strong continuity, however, implies that for every $x \in E$, $\sup_n \|T_{t_n} x\|_E$ is bounded. Thus by the Uniform Boundedness Principle we reach a contradiction. Now every $t > 0$ is of the form $t = Na + \delta$ for some $\delta < a$, hence by the semigroup property,

$$\|T_t\| = \|T_a^N T_\delta\| \leq M^N \cdot M$$

using that the operator norm of the composition is bounded by the product of the norms. Now we set $k = \frac{1}{a} \log M$, so that

$$e^{kt} = e^{\frac{t}{a} \log M} = M^{\frac{t}{a}} \geq M^N$$

as $t \geq Na$, justifying the bound. □

Exercise 7.5

Let T_t be the semigroup induced by a Markov Process whose transition function is absolutely continuous with respect to the Lebesgue Measure. Prove that T_t is not strongly continuous on $\mathcal{B}_b(\mathbb{R})$.

Proof. Take $f(y) = \mathbb{1}_{\{0\}}(y)$, then $T_t f = 0$ for all $t > 0$, but as $f(0) = 1$, then $\|T_t f - f\|_\infty = 1$ for all $t > 0$, so T_t is not strongly continuous. □

Exercise 7.6

Define the heat kernel $p : (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty)$ by

$$p_t(x, y) = \frac{1}{(2\pi t)^{\frac{d}{2}}} e^{-\frac{|x-y|^2}{2t}},$$

the associated transition function

$$P_t(x,A) = \int_A p_t(x,y)dy$$

and furthermore the heat semigroup

$$T_t f(x) = \int_{\mathbb{R}^d} f(y) P_t(x,dy).$$

1. Let W be a d-dimensional Brownian Motion. For $0 < s < t$ and $f \in \mathcal{B}_b(\mathbb{R}^d)$, write down an explicit expression for $\mathbb{E}\left[f(W_t)|W_s = x\right]$.
2. Verify that T_t is Feller.
3. You are given that for $p \geq 1$ every $f \in L^p(\mathbb{R}^d)$ and $t > 0$, $\|T_t f\|_{L^p} \leq \|f\|_{L^p}$. Prove that for $f \in L^p(\mathbb{R}^d)$, then
$$\lim_{t\to 0} \|T_t f - f\|_{L^p} = 0.$$
4. Determine the generator of T_t.

Proof. We take $d = 1$ for simplicity.

1. Recall firstly that P is the transition function for Brownian Motion, and from the definition of the transition function,
$$\mathbb{E}\left[f(W_t)|W_s\right] = \int_{\mathbb{R}} f(y) P_{t-s}(W_s, dy) = \int_{\mathbb{R}}\int_{\mathbb{R}} f(y) P_{t-s}(z,dy)\mu(dz)$$
where μ is the law of W_s. Thus, $\mathbb{E}\left[f(W_t)|W_s = x\right]$ is as above where $\mu = \delta_x$, hence
$$\mathbb{E}\left[f(W_t)|W_s = x\right] = \int_{\mathbb{R}} f(y) P_{t-s}(x,dy) = T_{t-s} f(x).$$

2. We have that
$$T_t f(x) = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{\frac{-(x-y)^2}{2t}} f(y) dy = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{\frac{-(z)^2}{2t}} f(z+x) dz$$
which is explicitly shown to be continuous by taking $x_k \to x$ and applying the Dominated Convergence Theorem using $\|f\|_\infty < \infty$.

3. Firstly we consider g smooth and compactly supported. Note that we can write
$$g(x) = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{\frac{-(z)^2}{2t}} g(x) dz$$
so
$$T_t g(x) - g(x) = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{\frac{-(z)^2}{2t}} \left(g(x+z) - g(x)\right) dz$$
from the calculation in the previous part. Choosing $\delta > 0$ such that for all $|z| < \delta$, $|g(x+z) - g(x)| < \varepsilon$,
$$T_t g(x) - g(x) \leq \varepsilon + \frac{2\|g\|_\infty}{\sqrt{2\pi t}} \int_{|z|\geq\delta} e^{\frac{-(z)^2}{2t}} dz$$

where the second term goes to zero as $t \to \infty$. In particular,

$$\|T_t g - g\|_\infty \to 0$$

as $t \to$, which implies convergence in L^p by the Dominated Convergence Theorem as $T_t g - g$ is uniformly bounded in L^p. To show the result we take a sequence of smooth compactly supported (g_n) such that $g_n \to f$ in L^p. Then

$$\|T_t f - f\|_{L^p} \leq \|T_t f - T_t g_n\|_{L^p} + \|T_t g_n - g_n\|_{L^p} + \|g_n - f\|_{L^p}.$$

In the first term we use that

$$\|T_t f - T_t g_n\|_{L^p} = \|T_t(f - g_n)\|_{L^p} \leq \|f - g_n\|_{L^p}$$

so that now we can take the limit as $t \to 0$ followed by the limit as $n \to \infty$ to see that the right hand side goes to zero as required.

4. As before, we simply show the generator as a pointwise limit. Using the first part, and taking $s = 0$ for notational simplicity (that is, changing the initial value of our Brownian Motion), we are considering

$$\lim_{t \to 0} \frac{T_t f(x) - f(x)}{t} = \lim_{t \to 0} \frac{\mathbb{E}\left[f(W_t) | W_0 = x\right] - f(x)}{t}$$

where we assume that f is smooth. Taking a Taylor Expansion about $f(x)$ gives us that

$$\mathbb{E}\left[f(W_t)|W_0 = x\right] = \mathbb{E}\left[f(x) + (W_t - x)f'(x) + \frac{1}{2}(W_t - x)^2 f''x + \cdots ....|W_0 = x\right]$$

where the remaining terms are of higher order in t, so will vanish in the limit as $t \to 0$. Computing these terms gives us

$$f(x) + \frac{t}{2} f''(x)$$

so at least formally,

$$\lim_{t \to 0} \frac{\mathbb{E}\left[f(W_t)|W_0 = x\right] - f(x)}{t} = \frac{1}{2} f''(x)$$

so we identify the generator as $\frac{1}{2}\Delta$.

□